STABILITY OF A PLANE-PARALLEL CONVECTIVE FLOW OF A LIQUID IN A HORIZONTAL LAYER

G. Z. Gershuni, E. M. Zhukhovitskii,
and V. M. Myznikov

UDC 532.516

We consider the stationary plane-parallel convective flow, studied in [1], which appears in a two-dimensional horizontal layer of a liquid in the presence of a longitudinal temperature gradient. In the present paper we examine the stability of this flow relative to small perturbations. To solve the spectral amplitude problem and to determine the stability boundaries we apply a version of the Galerkin method, which was used earlier for studying the stability of convective flows in vertical and inclined layers in the presence of a transverse temperature difference or of internal heat sources (see [2]). A horizontal plane-parallel flow is found to be unstable relative to two critical modes of perturbations. For small Prandtl numbers the instability has a hydrodynamic character and is associated with the development of vortices on the boundary of counterflows. For moderate and for large Prandtl numbers the instability has a Rayleigh character and is due to a thermal stratification arising in the stationary flow.

1. Stationary Flow. We consider a two-dimensional horizontal layer of a liquid, bounded by the solid planes $x = \pm h$. On the two planes the temperature is given and varies linearly with the coordinate z:

$$T_0 = Az \tag{1.1}$$

In a sufficiently long layer a plane-parallel stationary flow appears, having the following structure:

$$\begin{gathered} v_x = v_y = 0,\ v_z = v_0(x), \\ T_0 = Az + \tau_0(x),\ p = p_0(x, z) \end{gathered} \tag{1.2}$$

The stationary velocity, temperature, and pressure distributions may be obtained from the equations

$$\frac{1}{\rho}\frac{\partial p_0}{\partial x} = g\beta(Az + \tau_0),\ \frac{1}{\rho}\frac{\partial p_0}{\partial z} = \nu v_0'',\qquad Av_0 = \chi\tau_0'' \tag{1.3}$$

Here ρ is the average density, g is the gravitational acceleration, and ν, χ, and β are, respectively, the coefficients of kinematic viscosity, thermal diffusivity, and thermal expansion. On the boundaries of the layer we have

$$v_0 = 0,\ \tau_0 = 0 \quad (x = \pm h) \tag{1.4}$$

In addition, we assume the closed flow condition to be satisfied:

$$\int_{-h}^{h} v_0 dx = 0 \tag{1.5}$$

From the Eqs. (1.3), with the boundary conditions (1.4) and (1.5), we determine the stationary flow [1]

$$v_0 = \frac{g\beta Ah^3}{6\nu}\left[\left(\frac{x}{h}\right)^3 - \left(\frac{x}{h}\right)\right], \tag{1.6}$$

Perm'. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 95-100, January-February, 1974. Original article submitted June 4, 1973.

$$\tau_0 = \frac{g\beta A^2 h^5}{360\nu\chi}\left[3\left(\frac{x}{h}\right)^5 - 10\left(\frac{x}{h}\right)^3 + 7\left(\frac{x}{h}\right)\right] \quad p_0 = \rho g\beta\left[Axz + \int \tau_0 dx\right]$$

The flow consists of two horizontal counter flows. The form of the velocity distribution turns out to be of the kind which appears in the case of a vertical layer, the boundaries of which are maintained at different temperatures. It can be expected that for a sufficiently large pressure intensity (i.e., for a sufficiently large gradient A) a hydrodynamic type of instability will arise.

From the distribution of temperature it follows that, although for a given z there is no transverse temperature difference between the layer boundaries, the flow leads to the formation of two layers in the liquid, in the interior of which there is a potentially unstable temperature stratification. These layers are located close to the upper and lower boundary planes. For a sufficiently large vertical temperature difference in these layers (proportional to A^2) we can expect an instability of Rayleigh type to appear.

2. Equations for the Perturbations. Method of Solution. To study the stability of the stationary regime (1.6), we consider the perturbed flow $\mathbf{v}_0+\mathbf{v}$, T_0+T, p_0+p, where $\mathbf{v}$, T, and p are small perturbations of the plane-parallel flow. We introduce the following units of measurement for distance, time, velocity, temperature, and pressure: h, h^2/ν, $g\beta Ah^3/\nu$, Ah and $\rho g\beta Ah^2$. In dimensionless form the equations for small perturbations are

$$\partial \mathbf{v} / \partial t + G\left[(\mathbf{v}\nabla)\mathbf{v}_0 + (\mathbf{v}_0\nabla)\mathbf{v}\right] = -\nabla p + \Delta\mathbf{v} + T\boldsymbol{\gamma} \tag{2.1}$$

$$\partial T / \partial t + G\left[\mathbf{v}\nabla T_0 + \mathbf{v}_0\nabla T\right] = P^{-1}\Delta T \tag{2.2}$$

$$\operatorname{div}\mathbf{v} = 0 \tag{2.3}$$

Here γ is a unit vector, directed vertically upwards, $G = g\beta Ah^4/\nu^2$ is the Grashof number, $P = \nu/\chi$ is the Prandtl number, and v_0 and T_0 are the dimensionless unperturbed velocity and temperature profiles

$$v_0 = 1/6\,(x^3 - x), \quad T_0 = z + GP\tau_0, \qquad \tau_0 = 1/360\,(3x^5 - 10x^3 + 7x) \tag{2.4}$$

The perturbations satisfy the homogeneous boundary conditions

$$\mathbf{v} = 0, \quad T = 0 \quad (x = \pm 1) \tag{2.5}$$

We consider two-dimensional perturbations. In this case the velocity components v_x and v_z are different from zero and all the quantities are independent of the coordinate y. Introducing "normal" perturbations of the stream function and of the temperature, namely,

$$\psi = \varphi(x)\exp(-\lambda t + ikz), \quad T = \theta(x)\exp(-\lambda t + ikz) \tag{2.6}$$

we obtain for the amplitudes φ and θ the spectral problem

$$\Delta\Delta\varphi - ikG\,(v_0\Delta\varphi - v_0''\varphi) - ik\theta = -\lambda\Delta\varphi \quad (\Delta \equiv d^2/dx^2 - k^2) \tag{2.7}$$

$$P^{-1}\Delta\theta - ikG\,(v_0\theta - GP\tau_0'\varphi) - G\varphi' = -\lambda\theta \tag{2.8}$$

$$\varphi = \varphi' = 0, \quad \theta = 0 \quad (x = \pm 1) \tag{2.9}$$

The decrement λ, depending on the parameters G, P, and the wave number k, is a characteristic number of the boundary-value problem (2.7)-(2.9).

We note here the differences between the boundary-value problem (2.7)-(2.9) and the problem which arises in studying the stability of a stationary convective flow between vertical planes, heated to a different temperature [2]. One of these differences is that in the heat conduction Eq. (2.8) a new term, $G\varphi'$, is present; this term describes the convective heat transfer in the field of the longitudinally unperturbed temperature gradient. The second difference is in the form of the lift force term in the equation of motion (2.7); in the case considered the lift force is perpendicular to the planes. Another difference is that the unperturbed temperature has a more involved distribution over a cross-section.

Just as in the case of a flow between parallel plates heated to different temperatures [3-6], we solve the amplitude problem by applying Galerkin's method. We write the amplitudes φ and θ in the form of the expansions

$$\varphi = \sum_{n=0}^{N-1} a_n \varphi_n, \quad \theta = \sum_{m=0}^{M-1} b_m \theta_m \tag{2.10}$$

As the basis functions φ_n and θ_m, we use the normalized amplitudes of the perturbations in the fixed layer of liquid; these amplitudes are characteristic functions of the boundary-value problems

$$\begin{aligned} \Delta\Delta\varphi_n &= -\mu_n \Delta\varphi_n, \; \varphi_n(\pm 1) = \varphi_n'(\pm 1) = 0 \\ P^{-1}\Delta\theta_m &= -\nu_m \theta_m, \; \theta_m(\pm 1) = 0 \end{aligned} \tag{2.11}$$

In the approximations (2.10) we retained 10-20 of the functions appearing in the expansions of φ and θ. A standard procedure leads to a system of linear homogeneous equations for the coefficients a_n and b_m; the characteristic decrements $\lambda = \lambda(G, P, k)$ are obtained from the condition for the solvability of this system.

3. Perturbation Spectra and Stability Boundaries. Depending on the value of the Prandtl number, the flow instability is stipulated by two qualitatively different mechanisms.

For small values of the Prandtl number, as well as in the limiting case $P \to 0$, the instability is associated with hydrodynamic perturbations of monotonic type. Figure 1 presents the spectrum of decrements for fixed values of $P = 0.1$ and $k = 1.3$. Shown plotted are the real parts of the decrements λ_r as functions of the Grashof number. The dashed curves refer to branches of hydrodynamic type (μ levels), while the solid curves refer to thermal branches (ν levels). The dash-dot curves denote common real parts of complex-conjugate decrements, formed from the merging of a pair of real levels.

At points where the real branches merge, pairs of oscillating perturbations are formed. Their phase velocity (in units of maximum velocity of the unperturbed flow) is connected with the imaginary part of the decrement λ_i through the relationship $c = 9\sqrt{3}\lambda_i/kG$. As G increases, the phase velocity increases monotonically; for example, when $G = 1000$, the phase velocity of the oscillating perturbations formed from the merging of the real levels μ_1 and μ_2 has a value $c \approx 0.5$.

The lower level μ_0, which stays real, changes its sign at the critical value of the Grashof number; the flow becomes unstable relative to a perturbation possessing a zero phase velocity. By varying the parameter k, we can obtain the neutral stability curve G(k). Figure 2 presents a family of neutral curves of monotonic instability for certain values of P (curves 1, 2, 3, 4, and 5 refer, respectively, to the values of P = 0.01, 0.05, 0.1, 0.125, 0.15). When $P = 0$ (see [3, 4]), the minimum critical Grashof number $G_m = 495$. It can be seen that as P increases the stability is increased. Moreover, the critical wave number k_m decreases insignificantly.

A numerical study of the form of the neutral perturbations for small P shows that the total motion, formed from the imposition of a perturbation on the fundamental flow, constitutes a system of fixed vortices, periodic along the z axis, on the boundary of the counter flows.

The stabilizing effect with an increase in P is associated with the fact that for finite P in the layer there is a stable temperature stratification in the central part of the channel. Heating from above makes the development of vortices difficult. An analogous stabilizing effect was noted in a study of the stability of a Poiseuille flow in a horizontal channel heated from above [7, 8].

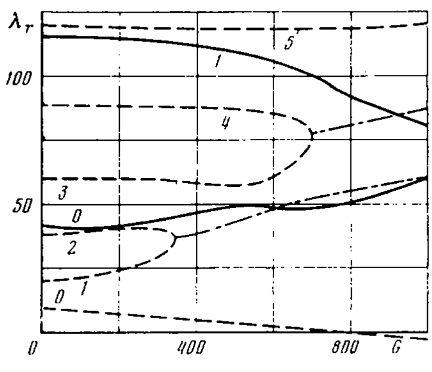

Fig. 1

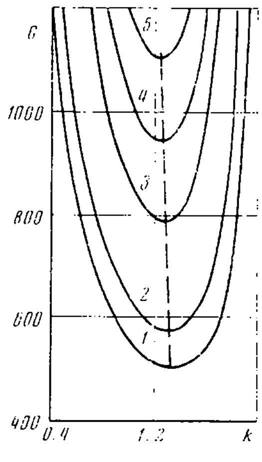

Fig. 2

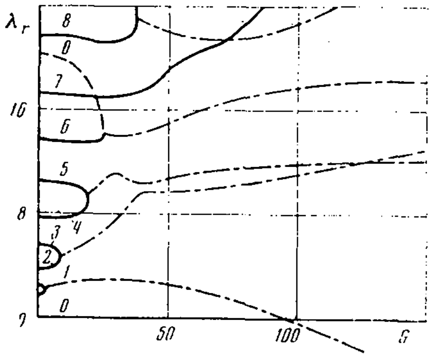

Fig. 3

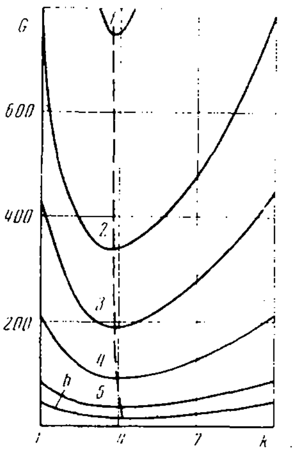

Fig. 4

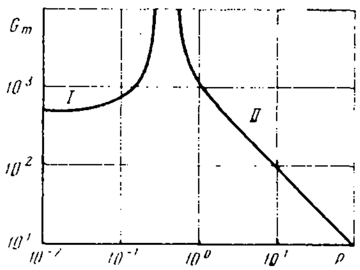

Fig. 5

The flow instability which arises for moderate and large Prandtl numbers is qualitatively of a different nature. An example of a spectrum of decrements is shown in Fig. 3 ($P = 10$, $k = 4$). The instability has an oscillatory character and is generated by a complex-conjugate pair of decrements formed in the merging of the thermal levels ν_0 and ν_1. At the critical point a pair of increasing perturbations arises, which differs by the sign of the phase velocity. The perturbation with the phase velocity $c < 0$ propagates in the form of a wave along the upper (relative to the more strongly heated) flow, while the perturbation with $c > 0$ is carried away by the lower (colder) flow.

The neutral curves G(k) are shown in Fig. 4; Curves 1, 2, 3, 4, 5 and 6 correspond to the Prandtl numbers $P = 1.5$, 3, 5, 10, 25, and 50. The most "dangerous" perturbations are the short-wave perturbations with $k_m \approx 4$. With increasing P, the minimum critical Grashof number G_m decreases, and for large P we have the asymptotic relationship

$$G_m = 964 / P \tag{3.1}$$

In this region of Prandtl numbers the stability boundary is determined by the critical Rayleigh number $R_m = G_m P = 964$.

The neutral critical perturbations have an oscillating character. Their phase velocity c_m (at a neutral curve minimum) depends weakly on P. As P increases from 0.6 to 50, c_m grows monotonically from 0.67 to 0.86.

Perturbations resulting in an instability of the type discussed have a cellular structure and are characterized by a definite localization in the flow. Perturbations with a negative phase velocity are localized in the upper part of the layer, where the region of unstable temperature stratification is located. These perturbations lead to the formation of waves propagating along the upper flow; the lower flow is practically unperturbed. Perturbations with a positive phase velocity constitute cells, concentrated in the unstably stratified lower layer, and their imposition on the main flow results in waves propagating along the lower flow.

The results given here confirm the fact that for large values of the Prandtl number P the instability is associated with the presence in the flow of potentially unstable zones for the distribution of temperature. Cellular perturbations arising in these zones are carried away by the main flow. The critical situation for the flow is determined by the critical Rayleigh number (3.1). As a characteristic of the flow, we can introduce, along with the Rayleigh number R determined in accord with the longitudinal gradient, also a Rayleigh number R_*, determined according to the transverse temperature difference on the boundaries of the (upper or lower) unstably stratified layer and its thickness. It is evident from the form of the unperturbed temperature profile (1.6) that these Rayleigh numbers are related as follows: $R_* = \text{const}\ R^2$. The formula (3.1) means that R_* is a critical parameter, serving as a typical criterion of Rayleigh instability. Localization of the perturbations in one of the unstably stratified layers is typical. With it there is associated a characteristic wave length for the cells which arise ($k_m \approx 4$), being found to be of a thickness on the order of a stratified layer.

The composite results, which relate to the stability boundaries, are shown in Fig. 5. Curve I shows the dependence of the minimum critical Grashof number G_m on the Prandtl number P for monotonic perturbations of a hydrodynamic character. Curve II relates to the Rayleigh instability mechanism.

The results presented here refer to the case of two-dimensional perturbations. For the problem considered there are no transformations, which make it possible to reduce a three-dimensional problem to a two-dimensional one (this is in contrast to the case of a layer with a transverse temperature difference, for which such transformations are available; see [9]). The problem concerning the behavior of three-dimensional perturbations requires a separate treatment.

LITERATURE CITED

1. R. V. Birikh, "On thermocapillary convection in a horizontal layer of a liquid," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 3 (1966).
2. G. Z. Gershuni and E. M. Zhukhovitskii, Convective Stability of an Incompressible Liquid [in Russian], Nauka, Moscow (1972).
3. R. V. Birikh, "On small perturbations of a plane-parallel flow with a cubic velocity profile," Prikl. Matem. i Mekhan., 30, No. 2 (1966).
4. R. N. Rudakov, "Perturbation spectrum and stability of convective motion between vertical planes," Prikl. Matem. i Mekhan., 31, No. 2 (1967).
5. R. V. Birikh, G. Z. Gershuni, E. M. Zhukhovitskii, and R. N. Rudakov, "Hydrodynamic and thermal instability of a stationary convective motion," Prikl. Matem. i Mekhan., 32, No. 2 (1968).
6. R. V. Birikh, G. Z. Gershuni, E. M. Zhukhovitskii, and R. N. Rudakov, "Stability of a stationary convective motion of a liquid with a longitudinal temperature gradient," Prikl. Matem. i Mekhan., 33, No. 6 (1969).
7. K. S. Gage and W. H. Reid, "The stability of thermally stratified plane Poiseuille flow," J. Fluid Mech., 33, Part 1 (1968).
8. K. S. Gage, "The effect of stable thermal stratification on the stability of viscous parallel flows," J. Fluid Mech., 47, Part 1 (1971).
9. G. Z. Gershuni and E. M. Zhukhovitskii, "On the stability of a plane-parallel convective motion relative to three-dimensional perturbations," Prikl. Matem. i Mekhan., 33, No. 5 (1969).